Composition of simplicial complexes, polytopes and multigraded Betti numbers

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Abstract. For a simplicial complex K on m vertices and simplicial complexes $K_1, \ldots, K_m$ a composed simplicial complex $K(K_1, \ldots, K_m)$ is introduced. This construction generalizes an iterated simplicial wedge construction studied by A. Bahri, M. Bendersky, F. R. Cohen and S. Gitler and allows to describe the combinatorics of generalized joins of polytopes $P(P_1, \ldots, P_m)$ defined by G. Agnarsson in most important cases. The composition defines a structure of an operad on a set of finite simplicial complexes, in which a complex on m vertices is viewed as an m-adic operation. We prove the following: (1) a composed complex $K(K_1, \ldots, K_m)$ is a simplicial sphere iff K is a simplicial sphere and K_i are the boundaries of simplices; (2) a class of spherical nerve-complexes is closed under the operation of composition (3) finally, we express multigraded Betti numbers of $K(K_1, \ldots, K_m)$ in terms of multigraded Betti numbers of $K, K_1, \ldots, K_m$ using a composition of generating functions.

1. Introduction

In toric topology multiple connections between convex polytopes, simplicial complexes, topological spaces and Stanley–Reisner algebras are studied. Starting with a simple polytope P one constructs a moment-angle manifold $\mathcal{Z}_P$ with a torus action such that its orbit space is the polytope P itself. On the other hand, a simplicial complex ∂P^* gives rise to a moment-angle complex $\mathcal{Z}_{\partial P^*}(D^2, S^1)$. This complex is homeomorphic to $\mathcal{Z}_P$ and possesses a natural cellular structure which allows to describe its cohomology ring: $H^*(\mathcal{Z}_P; \Bbbk) \cong \mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[\partial P^*], \Bbbk)$. This consideration can be used to translate topological problems to the language of Stanley–Reisner algebras and vice-versa. Moreover, the cohomology ring $H^*(\mathcal{Z}_P; \Bbbk)$ carries an information about the combinatorics of the polytope P from which we started.

With some modifications this setting can be generalized to nonsimple polytopes. If P is a convex polytope (possibly nonsimple), then the moment-angle space $\mathcal{Z}_P$ is defined as an intersection of real quadrics (but in nonsimple case $\mathcal{Z}_P$ is not a manifold). A simplicial complex K_P, called the nerve-complex [**3**], is associated to each polytope (in nonsimple case K_P is not a simplicial sphere). The complex K_P carries a complete information on the combinatorics of P and its properties are similar to simplicial spheres. Generally there is a homotopy equivalence $\mathcal{Z}_P \simeq \mathcal{Z}_{K_P}(D^2, S^1)$. An open question is to describe the properties of Stanley–Reisner algebras $\Bbbk[K_P]$ and cohomology rings $H^*(\mathcal{Z}_P; \Bbbk) \cong \mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K_P], \Bbbk)$ for nonsimple convex polytopes.

In the work of A. Bahri, M. Bendersky, F. R. Cohen and S. Gitler [**5**] a new construction is described, which allows to build a simple polytope $P(l_1,\ldots,l_m)$ from a given simple polytope P with m facets and an array $(l_1,\ldots,l_m)$ of natural numbers. A simplicial complex $\partial P(l_1,\ldots,l_m)^*$ can be described combinatorially in terms of missing faces. Such description gives a representation of $\mathcal{Z}_{\partial P(l_1,\ldots,l_m)^*}(D^2,S^1)$ as a polyhedral product $\mathcal{Z}_{P^*}(\underline{(D^{2l_i},S^{2l_i-1})})$ which leads, in particular, to alternative representation of the cohomology ring $H^*(\mathcal{Z}_{P(l_1,\ldots,l_m)})$.

The idea of treating nonsimple polytopes can be used to capture a wider class of examples and find more general constructions. One of the constructions is known in convex geometry (we refer to the work of Geir Agnarsson [**1**]). Given a polytope $P\subset\mathbb{R}^m_{\geqslant}$ and polytopes $P_1,\ldots,P_m$ a new polytope $P(P_1,\ldots,P_m)$ is constructed. This polytope generally depends on geometrical representation of $P\subset\mathbb{R}^m_{\geqslant}$, but under some restrictions the construction can be made combinatorial. In particular cases this construction gives the iterated polytope $P(l_1,\ldots,l_m)$ from the work [**5**]. Note, that the polytope $P(P_1,\ldots,P_m)$ may be nonsimple even in the case when all the polytopes $P,P_1,\ldots,P_m$ are simple.

In this work we introduce a new operation on the set of abstract simplicial complexes $K,K_1,\ldots,K_m\mapsto K(K_1,\ldots,K_m)$. This operation corresponds to the operation $P(P_1,\ldots,P_m)$ on convex polytopes and generalizes the constructions of [**5**]. The work is organized as follows:

(1) We review the construction of K_P and the definition of abstract spherical nerve-complex from the work [**3**]. Section 2.
(2) The construction of $P(P_1,\ldots,P_m)$. We give a few equivalent descriptions of this polytope and specialize the conditions under which $P(P_1,\ldots,P_m)$ is well defined on combinatorial polytopes. Section 3.
(3) Given a simplicial complex K on m vertices and simplicial complexes $K_1,\ldots,K_m$ we define a composed simplicial complex $K(K_1,\ldots,K_m)$, which is a central object of the work. Two equivalent definitions are provided: one is combinatorial, another describes $K(K_1,\ldots,K_m)$ as an analogue of polyhedral product called polyhedral join. It is shown that $K(\partial\Delta_{[l_1]},\ldots,\partial\Delta_{[l_m]})=K(l_1,\ldots,l_m)$ — an iterated simplicial wedge construction from the work [**5**]. We prove that $K_{P(P_1,\ldots,P_m)}=K_P(K_{P_1},\ldots,K_{P_m})$. Section 4.
(4) Polyhedral products defined by composed simplicial complexes. In section 5 we review and generalize some results from [**5**].
(5) In section 6 the structure of composed simplicial complexes is studied. At first we describe the homotopy type of $K(K_1,\ldots,K_m)$. It happens that $K(K_1,\ldots,K_m)\simeq K*K_1*\ldots*K_m$. The problem: for which choice of $K,K_1,\ldots,K_m$ the complex $K(K_1,\ldots,K_m)$ is a sphere? The answer: only in the case, when K is a sphere and $K_i=\partial\Delta_{[l_i]}$. Thus the class of simplicial spheres is not closed under the composition. Nevertheless, if $K,K_1,\ldots,K_m$ are spherical nerve-complexes, then so is $K(K_1,\ldots,K_m)$.
(6) In section 7 we describe the multigraded Betti numbers of $K(K_1,\ldots,K_m)$. There is a simple formula which expresses these numbers in terms of multigraded Betti numbers of $K,K_1,\ldots,K_m$. Applying this formula to $\partial\Delta_{[2]}(K_1,K_2)$ and $o^2(K_1,K_2)$, where o^2 is the complex with 2 ghost vertices, gives the result of [**3**]. Using the connection between bigraded Betti numbers and h-polynomial, found by V.M.Buchstaber and T.E.Panov [**7**], in section 8 we provide formulas for h-polynomials of compositions in some particular cases. Some of these formulas were found earlier by Yu.Ustonovsky [**16**].

The following notation and conventions are used. The simplicial complex K on a set of vertices $[m]$ is the system of subsets $K \subseteq 2^{[m]}$, such that $I \in K$ and $J \subset I$ implies $J \in K$. A vertex $i \in [m]$ such that $\{i\} \notin K$ is called ghost vertex. If $I \in K$, then $\operatorname{link}_K I$ is the simplicial complex on a set $[m] \setminus I$ such that $J \in \operatorname{link}_K I \Leftrightarrow J \sqcup I \in K$. Note that a link may have ghost vertices even if K does not have them. From the geometrical point of view the complex does not change when ghost vertices are omitted. We use the same symbol for the simplicial complex K and its geometrical realization. The complex K is called a simplicial sphere if it is PL-homeomorphic to the boundary of a simplex (we omit ghost vertices if necessary). Simplicial complex K is called a generalized homological sphere (or Gorenstein* complex) if K and all its links have homology of spheres of corresponding dimensions. If K is a simplicial sphere (resp. Gorenstein* complex) then so is $\operatorname{link}_K I$ for each $I \in K$.

If $A \subset [m]$, then full subcomplex K_A is the complex on A such that $J \in K_A \Leftrightarrow J \in K$. We denote the full simplex on the set $[m]$ by $\Delta_{[m]}$, it has dimension $m-1$. Its boundary $\partial\Delta_{[m]}$ — is complex on $[m]$, consisting of all proper subsets of $[m]$.

The notation $\bar{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ is used for arrays of numbers, and $\langle \bar{x}, \bar{y} \rangle$ denotes the sum $x_1y_1 + x_2y_2 + \ldots + x_my_m$. Sometimes double arrays will be used: $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m) = (x_{11}, \dots, x_{1l_1}, \dots, x_{m1}, \dots, x_{ml_m})$.

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2. Polytopes and nerve-complexes

Let P be an n-dimensional polytope and let $\{\mathcal{F}_1, \dots, \mathcal{F}_m\}$ be the set of all its facets. Consider a simplicial complex K_P on the set $[m] = \{1, \dots, m\}$ called the nerve-complex of a polytope P, defined by the condition $I = \{i_1, \dots, i_k\} \in K_P$ whenever $\mathcal{F}_{i_1} \cap \ldots \cap \mathcal{F}_{i_k} \neq \varnothing$. The complex K_P is thus the nerve of the closed covering of the boundary ∂P by facets.

Example 2.1. If P is simple, then K_P coincides with a boundary of a dual simplicial polytope: $K_P = \partial P^*$. In this case K_P is a simplicial sphere. It can be shown that K_P is not a sphere if P is not simple.

As shown in [**3**] nerve-complexes are nice substitutes for nonsimple polytopes. In particular, the moment-angle space $\mathcal{Z}_P$ of any convex polytope P is homotopy equivalent to the moment-angle complex $\mathcal{Z}_{K_P}(D^2, S^1)$, the Buchstaber numbers $s(P)$ and $s(K_P)$ are equal, etc.

There are necessary conditions on the complex K to be the nerve-complex of some convex polytope. These conditions are gathered in the notion of a **spherical nerve-complex**.

Let K be a simplicial complex, $M(K)$ — the set of its maximal (under inclusion) simplices. Let $F(K) = \{I \in K \mid I = \cap J_i, \text{ where } J_i \in M(K)\}$. The set $F(K)$ is partially ordered by inclusion. It can be shown (see [**3**]) that for each simplex $I \notin F(K)$ the complex $\operatorname{link}_K I$ is contractible.

Definition 2.2 (Spherical nerve-complex). *Simplicial complex K is called a spherical nerve-complex of rank n if the following conditions hold:*

- $\varnothing \in F(K)$, *i.e. intersection of all maximal simplices of K is empty;*

- *$F(K)$ is a graded poset of rank n (it means that all its saturated chains have the cardinality $n+1$). In this case the rank function* $\mathrm{rank}\colon F(K) \to \mathbb{Z}_{\geqslant}$ *is defined, such that* $\mathrm{rank}(I) =$ *the cardinality of saturated chain from $\varnothing$ to I minus* 1.
- *For any simplex $I \in F(K)$ the simplicial complex* $\mathrm{link}_K I$ *is homotopy equivalent to a sphere $S^{n-\mathrm{rank}(I)-1}$. Here, by definition,* $\mathrm{link}_K \varnothing = K$ *and $S^{-1} = \varnothing$.*

STATEMENT 2.3. *If P is an n-dimensional polytope, then K_P is a spherical nerve-complex of rank n and, moreover, the poset $F(K_P)$ is isomorphic to the poset of faces of P ordered by reverse inclusion.*

As a corollary, the poset of faces of P can be restored from K_P, thus K_P is a complete invariant of a combinatorial polytope P.

3. Composition of polytopes

Let $[m] = \{1, \ldots, m\}$ be a finite set and $\triangle_{[m]}$ be a standard $(m-1)$-dimensional simplex in $\mathbb{R}^m$ given by $\{\bar{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geqslant 0; \sum x_i = 1\}$. The convex polytope $P \subset \mathbb{R}^m$ will be called **stochastic** if $P \subseteq \triangle_{[m]}$. The following definition is due to [**1**, def.4.5].

DEFINITION 3.1. *Let $P \subseteq \mathbb{R}^m$ and $P_i \subseteq \mathbb{R}^{l_i}$ for $i \in [m]$ be stochastic polytopes. The polytope*

$$P(P_1, \ldots, P_m) = \{(t_1\bar{x}_1, t_2\bar{x}_2, \ldots, t_m\bar{x}_m) \in \mathbb{R}^{\sum l_i} \mid \\ \mid \bar{t} = (t_1, \ldots, t_m) \in P, \bar{x}_i \in P_i \text{ for each } i\} \tag{3.1}$$

is called the composition of polytopes P and $\{P_i\}$.

In [**1**] this operation is called the action of P.

EXAMPLE 3.2. $\triangle_{[m]}(P_1, \ldots, P_m) = P_1 * \ldots * P_m$ — the join of polytopes.

The original motivation of definition 3.1 was to extend the notion of the join to more general convex sets of parameters t_i.

REMARK 3.3. Definition 3.1 depends crucially on the geometrical representation of polytopes, not only their combinatorial type.

DEFINITION 3.4. *Let $L \subseteq \mathbb{R}^m$ be an affine n-dimensional subspace such that $P = L \cap \mathbb{R}^m_{\geqslant}$ is a nonempty bounded set (thus a polytope). If P is a stochastic polytope and every facet $\mathcal{F}_i \subset P$ is defined uniquely as $\mathcal{F}_i = P \cap \{x_i = 0\}$ we call P a* ***natural (stochastic) polytope****.*

REMARK 3.5. A natural stochastic polytope P in $\mathbb{R}^m$ has exactly m facets.

For a point $\bar{x} \in \mathbb{R}^m_{\geqslant}$ define $\hat{\sigma}(\bar{x}) = \{i \in [m] \mid x_i = 0\}$.

REMARK 3.6. For a natural stochastic polytope $P \subseteq \mathbb{R}^m_{\geqslant}$ the nerve-complex can be defined by the condition: $I \in K_P$, whenever there exists a point $\bar{x} \in P$ such that $I \subseteq \hat{\sigma}(\bar{x})$. Indeed, $I \in K_P$ implies that $\bigcap_{i \in I} \mathcal{F}_i \neq \varnothing$. Let $\bar{x} \in \bigcap_{i \in I} \mathcal{F}_i$. Then $x_i = 0$ for each $i \in I$ therefore $I \in \hat{\sigma}(\bar{x})$.

OBSERVATION 3.7. *Any polytope P is affine equivalent to a natural stochastic polytope.*

Observation 3.8. *The space $L \subseteq \mathbb{R}^m$ in the definition 3.4 can be defined by the system of affine relations $L = \{\bar{x} \in \mathbb{R}^m \mid \sum_j c_i^j x_j + d_i = 0$ for $i = 1, \ldots, m-n\}$ where all the coefficients c_i^j are positive and $d_i = -1$.*

Proof of both observations. Let

$$P = \{\bar{y} \in \mathbb{R}^n \mid \langle \bar{a}_i, \bar{y} \rangle + b_i \geqslant 0, i \in [m]\} \tag{3.2}$$

be a representation of P as an intersection of halfspaces, where $\bar{a}_i$ is the inner normal vector to the i-th facet (we suppose that there are no excess inequalities in (3.2) and $|\bar{a}_i| = 1$).

Consider an affine embedding $j_P \colon \mathbb{R}^n \to \mathbb{R}^m$, given by $j_P(\bar{y}) = (\langle \bar{a}_1, \bar{y} \rangle + b_1, \ldots, \langle \bar{a}_m, \bar{y} \rangle + b_m)$. Obviously, $j_P(P) \subseteq \mathbb{R}^m_\geqslant$ and, moreover, $j_P(P) = j_P(\mathbb{R}^n) \cap \mathbb{R}^m_\geqslant$. Denote the affine subspace $j_P(\mathbb{R}^n)$ by L. This subspace is given by the system of affine relations $L = \{\bar{x} \in \mathbb{R}^m \mid \langle \bar{c}_i, \bar{x} \rangle + d_i = 0 \text{ for } i = 1, \ldots, m-n\}$. The facets of $j_P(P)$ are given by $j_P(P) \cap \{x_i = 0\}$.

Notice that there is a relation $\sum S_i \bar{a}_i = 0$ by Minkowski theorem, where $S_i > 0$ are the $(n-1)$-volumes of facets. Then one of the affine relations for L has the form $\sum_i S_i x_i + d = 0$ with all the coefficients S_i strictly positive. Adding this relation multiplied by large enough number to other relations leads to a system of relations with positive coefficients.

Now divide each relation by d_i to get the relations of the form $\sum c_i^j x_j = 1$. Set new variables $x'_j = c_1^j x_j$ to transform one of the relations to the form $\sum x_j = 1$. This gives a stochastic polytope in $\mathbb{R}^m$. Observations proved. □

Proposition 3.9. *Let $P \in \mathbb{R}^m$ be a natural stochastic polytope given by $P = \mathbb{R}^m_\geqslant \cap \{\langle \bar{c}_i, \bar{x} \rangle = 1, i = 1, \ldots, m-n\}$, $\bar{c}_i = (c_i^1, \ldots, c_i^m)$ and for each $i \in [m]$ a natural stochastic polytope $P_i \in \mathbb{R}^{l_i}$ is given by $P_i = \mathbb{R}^{l_i}_\geqslant \cap \{\langle \bar{c}_{ij_i}, \bar{x}_i \rangle = 1, j_i = 1, \ldots, l_i - n_i\}$, $\bar{c}_{ij_i} = (c^1_{ij_i}, \ldots, c^{l_i}_{ij_i})$. Then the polytope $P(P_1, \ldots, P_m)$ is a natural stochastic polytope described by the system*

$$\begin{aligned} P(P_1, \ldots, P_m) = \{(\bar{x}_1, \ldots, \bar{x}_m) \in \mathbb{R}^{l_1}_\geqslant \times \ldots \times \mathbb{R}^{l_m}_\geqslant = \mathbb{R}^{\sum l_i}_\geqslant \mid \\ \mid c_i^1 \langle \bar{c}_{1j_1}, \bar{x}_1 \rangle + c_i^2 \langle \bar{c}_{2j_2}, \bar{x}_2 \rangle + \ldots + c_i^m \langle \bar{c}_{mj_m}, \bar{x}_m \rangle = 1\} \end{aligned} \tag{3.3}$$

Proof. By direct substitution $P(P_1, \ldots, P_m)$ as defined in 3.1 satisfies all the specified affine relations. On the contrary let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \in \mathbb{R}^{\sum l_i}_\geqslant$ satisfies relations (3.3) for all $i, j_1, \ldots, j_m$. Denote $\langle \bar{c}_{ij_i}, \bar{x}_i \rangle \in \mathbb{R}$ by $t_i(j_i)$. Then $t_i(j_i) \geqslant 0$ (by nonnegativity of coefficients in affine relations) and $c_i^1 t_1(j_1) + c_i^2 t_2(j_2) + \ldots + c_i^m t_m(j_m) = 1$ for each i, therefore $\bar{t}(\bar{j}) = (t_1(j_1), \ldots, t_m(j_m)) \in P$.

Let us show that $t_i(j_i)$ does not actually depend on j_i. Consider first entry j_1 for simplicity. Let j_1 and j'_1 be different indices. The point $\bar{x}$ satisfies the relations

$$c_i^1 \langle \bar{c}_{1j_1}, \bar{x}_1 \rangle + c_i^2 \langle \bar{c}_{2j_2}, \bar{x}_2 \rangle + \ldots + c_i^m \langle \bar{c}_{mj_m}, \bar{x}_m \rangle = 1$$

and

$$c_i^1 \langle \bar{c}_{1j'_1}, \bar{x}_1 \rangle + c_i^2 \langle \bar{c}_{2j_2}, \bar{x}_2 \rangle + \ldots + c_i^m \langle \bar{c}_{mj_m}, \bar{x}_m \rangle = 1$$

Subtracting we get $c_i^1 t_1(j_1) = c_i^1 \langle \bar{c}_{1j_1}, \bar{x}_1 \rangle = c_i^1 \langle \bar{c}_{1j_1}, \bar{x}_1 \rangle = c_i^1 t_1(j'_1)$. Since $c_i^1 \neq 0$ (at least for one i) we get $t_1(j_1) = t_1(j'_1)$.

Thus far we can simply write t_i instead of $t_i(j_i)$. Then $\bar{t} = (t_1, \ldots, t_m) \in P$. As a consequence, $\langle \bar{c}_{ij_i}, \frac{\bar{x}_i}{t_i} \rangle = 1$ for each i and j_i. Then $\bar{x} = (t_1 \frac{\bar{x}_1}{t_1}, t_2 \frac{\bar{x}_2}{t_2}, \ldots, t_m \frac{\bar{x}_m}{t_m})$ where $\bar{t} \in P$ and $\frac{\bar{x}_i}{t_i} \in P_i$. This means $\bar{x} \in P(P_1, \ldots, P_m)$ by definition. □

Example 3.10. Let $P \subseteq \mathbb{R}^m_{\geqslant}$ be a natural stochastic polytope (with m facets) defined by relations $\{\langle \bar{c}_i, \bar{x} \rangle = 1\}$ and $\Delta_{[l_i]} \subseteq \mathbb{R}^{l_i}$ a standard simplex given by $\{x_1 + \ldots + x_{l_i} = 1\}$. The polytope $P(l_1, \ldots, l_m) = P(\triangle_{[l_1]}, \ldots, \triangle_{[l_m]}) \subseteq \mathbb{R}^{\sum l_i}$ is called the **iteration** of the polytope P. It is given in $\mathbb{R}^{\sum l_i}_{\geqslant}$ by the system of affine relations

$$c_i^1(x_{11} + \ldots + x_{1l_1}) + c_i^2(x_{21} + \ldots + x_{2l_2}) + \ldots + c_i^m(x_{m1} + \ldots + x_{ml_m}) = 1. \tag{3.4}$$

If P is simple then $P(l_1, \ldots, l_m)$ is simple as well (see section 6 or the work **[5]**). Such polytopes, their quasitoric manifolds and moment-angle complexes were studied in **[5]**. For the particular case $P(l, \ldots, l)$, $l > 0$ we use the notation lP.

Remark 3.11. In section 6 we will show that for natural stochastic polytopes the operation $P(P_1, \ldots, P_m)$ depends up to combinatorial equivalence only on the combinatorial type of polytopes. Since each polytope has a natural stochastic representation we can view the composition as the operation on combinatorial polytopes.

Proposition 3.12 (Associativity law for the composition of polytopes). *Let P be a stochastic polytope in $\mathbb{R}^m_{\geqslant}$, $P_1, \ldots, P_m$ be stochastic polytopes in $\mathbb{R}^{l_1}_{\geqslant}, \ldots, \mathbb{R}^{l_m}_{\geqslant}$ respectively and*

$$P_{11}, \ldots, P_{1l_1}, P_{21}, \ldots, P_{2l_2}, \ldots, P_{m1}, \ldots, P_{ml_m}$$

— stochastic polytopes as well. Then

$$\begin{aligned} P(P_1(P_{11}, \ldots, P_{1l_1}), \ldots, P_m(P_{m1}, \ldots, P_{ml_m})) = \\ P(P_1, \ldots, P_m)(P_{11}, \ldots, P_{1l_1}, \ldots, P_{m1}, \ldots, P_{ml_m}). \end{aligned} \tag{3.5}$$

The proof follows easily from the definition 3.1.

Remark 3.13. It can be seen that $P(\mathrm{pt}, \ldots, \mathrm{pt}) = \mathrm{pt}(P) = P$, where $\mathrm{pt} = \triangle_{[1]}$ is a point. Thus far the set of all stochastic polytopes carries the structure of an operad, where the polytope in $\mathbb{R}^m_{\geqslant}$ is viewed as m-adic operation and the composition is given by the composition of polytopes described above. Proposition 3.12 expresses the associativity condition for the operad and the polytope pt is the “identity” element. Natural stochastic polytopes form a suboperad by proposition 3.9.

4. Composition of simplicial complexes

Consider a simplicial complex K on m vertices and a set of topological pairs $\{(X_i, A_i)\}_{i \in [m]}$, $A_i \subseteq X_i$. For a simplex $I \in K$ let V_I be the subset of $X_1 \times \ldots \times X_m$ given by $V_I = C_1 \times \ldots \times C_m$, where $C_i = X_i$ if $i \in I$ and $C_i = A_i$ if $i \notin I$. The space

$$\mathcal{Z}_K(\underline{(X_i, A_i)}) = \bigcup_{I \in K} V_I \subseteq \prod_i X_i$$

is called the **polyhedral product** of pairs (X_i, A_i) defined by K.

Example 4.1. The motivating examples of polyhedral products are moment-angle complexes $\mathcal{Z}_K(D^2, S^1)$, real moment-angle complexes $\mathcal{Z}_K(D^1, S^0)$ and Davis–Januszkiewicz spaces $DJ(K) = \mathcal{Z}_K(CP^\infty, \mathrm{pt})$ (see **[8]**). Another series of examples is given by wedges $\bigvee_\alpha X_\alpha \cong \mathcal{Z}_{\Delta^{(0)}_{[m]}}(\underline{(X_\alpha, \mathrm{pt})})$, fat wedges $\mathcal{Z}_{\partial\Delta_{[m]}}(\underline{(X_\alpha, \mathrm{pt})})$ and generalized fat wedges $\mathcal{Z}_{\Delta^{(k)}_{[m]}}(\underline{(X_\alpha, \mathrm{pt})})$. The spaces of the form $\mathcal{Z}_K(\underline{(X_\alpha, \mathrm{pt})})$ were studied in **[2]**. The most general situation

$\mathcal{Z}_K(\underline{(X_i, A_i)})$ was defined and studied by A. Bahri, M. Bendersky, F. R. Cohen and S. Gitler in **[4]** from the homotopy point of view.

The very natural thing is to substitute the topological product in the definition of a polyhedral product by any other operation on topological spaces. Thus far we can get **polyhedral smash product** $\mathcal{Z}_K^{\wedge}(\underline{(X_i, A_i)})$ **[4]** and **polyhedral join** $\mathcal{Z}_K^{*}(\underline{(X_i, A_i)})$ as defined below.

Definition 4.2. *Let* $\{(X_i, A_i)\}_{i\in[m]}$ *be topological pairs and* K *a simplicial complex on* $[m]$*. For each simplex* $I \in K$ *consider a subset* $V_I \subseteq X_1 * \ldots * X_m$ *of the form* $V_I = C_1 * \ldots * C_m$*, where* $C_i = X_i$ *if* $i \in I$ *and* $C_i = A_i$ *if* $i \notin I$*. The space*

$$\mathcal{Z}_K^{*}(\underline{(X_i, A_i)}) = \bigcup_{I\in K} V_I \subseteq \circledast X_i$$

is called the polyhedral join of pairs (X_i, A_i)*.*

Observation 4.3. *If* X_i *is a simplicial complex and* A_i *its simplicial subcomplex, the space* $\mathcal{Z}_K^{*}(\underline{(X_i, A_i)})$ *has a canonical simplicial structure. So far the polyhedral join is well defined on the category of simplicial complexes as opposed to polyhedral product.*

Let K be a simplicial complex on the set $[m]$. It can be considered as a subcomplex of $\Delta_{[m]}$ — the simplex on the set $[m]$, so far there is a pair $(\Delta_{[m]}, K)$.

Definition 4.4. *Let* K *be a simplicial complex on the set* $[m]$ *and* K_i *a simplicial complex on the set* $[l_i]$ *for each* $i \in [m]$*. The simplicial complex* $K(K_1, \ldots, K_m) = \mathcal{Z}_K^{*}(\underline{(\Delta_{[l_i]}, K_i)})$ *will be called the* ***composition*** *of* K *with* $K_i, i \in [m]$*.*

Now we define the composition of simplicial complexes in purely combinatorial terms. Let K be a simplicial complex on m vertices, which are possibly ghost. Let $K_1, \ldots, K_m$ be simplicial complexes on the sets $[l_1], \ldots, [l_m]$ (ghost vertices are allowed as well). Then $K(\underline{K_i})$ is a simplicial complex on the set $[l_1] \sqcup \ldots \sqcup [l_m]$ defined by the following condition: the set $I = I_1 \sqcup \ldots \sqcup I_m, I_i \subseteq [l_i]$ is the simplex of $K(K_1, \ldots, K_m)$ whenever $\{i \in [m] \mid I_i \notin K_i\} \in K$.

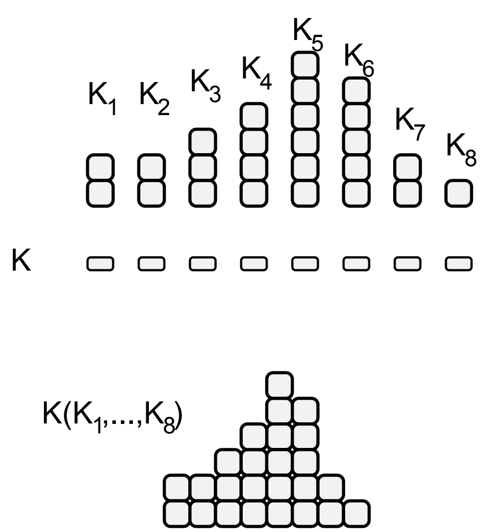

Figure 1. Vertices of $K(K_1, \ldots, K_m)$

The process of constructing the complex $K(\underline{K_\alpha}) = K(K_1, \ldots, K_m)$ is depicted on figures 1 and 2. The set of vertices of $K(\underline{K_\alpha})$ is the union of vertices of K_i, which can be depicted by a simple diagram (fig. 1). To construct the simplex of $K(\underline{K_\alpha})$ we fix any simplex $J \in K$ and take full subcomplex $\Delta_{[l_i]}$ (or any of its faces) for $i \in J$ and any simplex $I_i \in K_i$ for each $i \notin J$. The union of these sets gives a simplex of $K(\underline{K_\alpha})$ (fig. 2). All simplices $I \in K(\underline{K_\alpha})$ can be constructed by such procedure. This approach to the construction of $K(\underline{K_\alpha})$ will be discussed in section 6 in more detail.

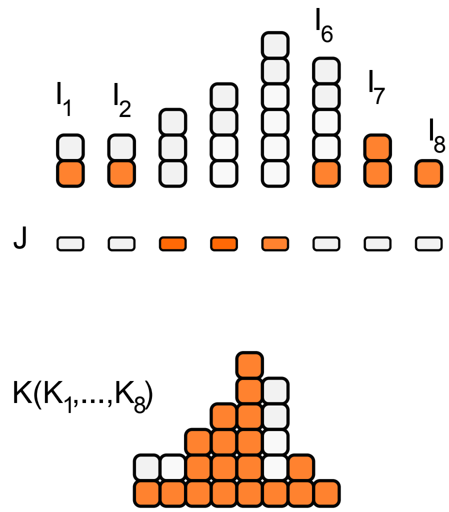

Figure 2. Simplex of $K(K_1, \ldots, K_m)$

Let o^l be the simplicial complex on $l > 0$ vertices which has no nonempty simplices. It means that all its vertices are ghost. We formally set $\partial\Delta_{[1]} = o^1$.

By remark 3.6 we can set $K_{\mathrm{pt}} = o^1$ since the polytope $\mathrm{pt} = \triangle_{[1]}$ is defined by $\mathbb{R}_{\geqslant} \cap \{x \in \mathbb{R} \mid x = 1\}$ and does not intersect the hyperplane $\{x = 0\}$.

Example 4.5. We have by definition $K(o^1, \ldots, o^1) = K$ and $o^1(K) = K$.

Example 4.6. The complex $K(o^l, o^1, \ldots, o^1)$. Let v_1 be the first vertex of K. Then $K(o^l, o, \ldots, o)$ can be described by the following procedure: the vertex v_1 is replaced by a simplex $I_1 = \{v_1^1, \ldots, v_1^l\}$ and simplices $I \in K$, containing v_1 are blown up to simplices $(I \setminus \{v_1\}) \sqcup I_1$. Therefore, $K(o^l, o, \ldots, o) = K_{[m]\setminus v_1} \cup (\mathrm{link}_K\, v_1) * I_1$.

Example 4.7. $o^m(K_1, \ldots, K_m) = K_1 * \ldots * K_m$.

Next statement provides a connection between the composition of polytopes (in the natural stochastic case) and the composition of simplicial complexes.

Proposition 4.8. *Let $P, P_1, \ldots, P_m$ be natural stochastic polytopes. Then*

$$K_{P(P_1,\ldots,P_m)} = K_P(K_{P_1}, \ldots, K_{P_m}).$$

Proof. We need a technical lemma. Recall from section 3 that for $\bar{x} \in \mathbb{R}^m_{\geqslant}$, $\bar{t}(\bar{x}) = \{i \in [m] \mid x_i = 0\}$.

LEMMA 4.9. *Let Q be a polytope given by $\mathbb{R}^m_{\geqslant} \cap \{\sum c_i^j x_j = 1, i = 1, \ldots, m-n\}$ with $c_i^j > 0$. Fix $y \in \mathbb{R}$. If $\bar{x} \in \mathbb{R}^m_{\geqslant}$ is the solution to the system of equations $\sum c_i^j x_j = y$, then $y \geqslant 0$ and either $\hat{\sigma}(\bar{x}) \in K_Q$ if $y > 0$ or $\bar{x} = \bar{0}$ if $y = 0$.*

PROOF. If $y = 0$, the statement is evident since $c_i^j > 0$ and $\bar{x}$ should be nonnegative. If $y > 0$ consider the point $\bar{x}/y$. It can be seen that $\bar{x}/y \in Q$ and $\hat{\sigma}(\bar{x}/y) = \hat{\sigma}(\bar{x})$. Therefore $\hat{\sigma}(\bar{x}) \in K_Q$. $\square$

Let $\bar{x} = (x_1, \ldots, x_l) \in \mathbb{R}^l$. If $\bar{x} \in P$, then $\hat{\sigma}(\bar{x}) \in K_P$. Vice-versa, if $I \in K_P$ then there exists $\bar{x} \in P$ such that $I \subseteq \hat{\sigma}(\bar{x})$ (remark 3.6).

It can be seen that both complexes $K_{P(P_1,\ldots,P_m)}$ and $K_P(K_{P_1}, \ldots, K_{P_m})$ have the same set of vertices $[l_1] \sqcup \ldots \sqcup [l_m]$. Denote $l_1 + \ldots + l_m$ by Σ. Let

$$\bar{x} = (x_{11}, \ldots, x_{1l_1}, x_{21}, \ldots, x_{2l_2}, \ldots, x_{m1}, \ldots, x_{ml_m}) \in \mathbb{R}^{\Sigma}_{\geqslant} = (\bar{x}_1, \ldots, \bar{x}_m)$$

be the point of $P(P_1, \ldots, P_m)$. Then for the point $\bar{x}$ of $P(P_1, \ldots, P_m)$ we have

$$\langle \bar{c}_i, (\langle \bar{c}_{1j_1}, \bar{x}_1 \rangle, \langle \bar{c}_{2j_2}, \bar{x}_2 \rangle, \ldots, \langle \bar{c}_{mj_m}, \bar{x}_m \rangle) \rangle = 1.$$

Denote $\langle \bar{c}_{sj_s}, \bar{x}_s \rangle$ by t_s (it does not depend on $j_1, \ldots, j_m$ — see proof of proposition 3.9) and set $\bar{t} = (t_1, \ldots, t_m)$. By observation 3.8 we may assume $t_s \geqslant 0$. Therefore, $\hat{\sigma}(\bar{t}) \in K_P$. For all $s \in [m]$ we have an alternative:

- If $s \in \hat{\sigma}(\bar{t})$, then $t_s = 0$ and $\langle \bar{c}_{sj_s}, \bar{x}_s \rangle = 0$. Then $\bar{x}_s = \bar{0}$ by lemma 4.9.
- If $s \notin \hat{\sigma}(\bar{t})$, then $t_s \neq 0$ and $\langle \bar{c}_{sj_s}, \bar{x}_s \rangle = t_s > 0$. Then by lemma 4.9 $\hat{\sigma}(\bar{x}_s) \in K_{P_s}$.

Therefore, $\hat{\sigma}(\bar{x}) \in K_P(K_{P_1}, \ldots, K_{P_m})$. Preceding arguments show that if $I \in K_{P(P_1,\ldots,P_m)}$, then $I \in K_P(K_{P_1}, \ldots, K_{P_m})$. Now let $J \in K_P(K_{P_1}, \ldots, K_{P_m})$, $J = A_1 \sqcup \ldots \sqcup A_m$, where $A_s \subseteq [l_s]$. We need to show that there exists a point $\bar{x} \in P(P_1, \ldots, P_m)$ such that $J \in \hat{\sigma}(\bar{x})$.

By definition there exists a simplex $I \in K_P$ such that $s \notin I$ implies $A_s \in K_{P_s}$. There exists a point $\bar{t} = (t_1, \ldots, t_m) \in P$ such that $I \subseteq \hat{\sigma}(\bar{t})$. Also for each s there exist solutions to the system of equations $\{\langle \bar{c}_{s,j_s}, \bar{x}_s \rangle = t_s\}_{j_s=1,\ldots,l_s}$ such that $A_s \subseteq \hat{\sigma}(\bar{x}_s)$ if $t_s \neq 0$ and $\bar{x}_s = \bar{0}$ if $t_s = 0$ (equiv. $\hat{\sigma}(\bar{x}_s) = [l_s] \supseteq A_s$). Then the nonnegative solution to the system of equations

$$\langle \bar{c}_i, (\langle \bar{c}_{1j_1}, \bar{x}_1 \rangle, \langle \bar{c}_{2j_2}, \bar{x}_2 \rangle), \ldots, \langle \bar{c}_{mj_m}, \bar{x}_m \rangle) \rangle = 1$$

is given by $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$, where $\hat{\sigma}(\bar{x}) = \hat{\sigma}(\bar{x}_1) \sqcup \ldots \sqcup \hat{\sigma}(\bar{x}_m) \supseteq J$. This concludes the proof. $\square$

COROLLARY 4.10. *If $P, P_1, \ldots, P_m$ are combinatorially equivalent to $Q, Q_1, \ldots, Q_m$ respectively and all the polytopes are natural stochastic, then $P(P_1, \ldots, P_m)$ is combinatorially equivalent to $Q(Q_1, \ldots, Q_m)$. Therefore, $P(P_1, \ldots, P_m)$ can be viewed as a well-defined operation on combinatorial polytopes.*

EXAMPLE 4.11. A nontrivial example of the composition is the **iterated simplicial wedge** construction as defined in **[5]**. Let K be a simplicial complex on m vertices and $(l_1, \ldots, l_m)$ — an array of natural numbers. Consider the simplicial complex $K(l_1, \ldots, l_m) = K(\partial\Delta_{[l_1]}, \ldots, \partial\Delta_{[l_m]})$.

If P is a polytope, then $K_P(l_1, \ldots, l_m) = K_P(\partial\Delta_{[l_1]}, \ldots, \partial\Delta_{[l_m]}) = K_P(K_{\triangle_{[l_1]}}, \ldots, K_{\triangle_{[l_m]}}) = K_{P(\triangle_{[l_1]},\ldots,\triangle_{[l_m]})} = K_{P(l_1,\ldots,l_m)}$ by proposition 4.8. In section 6 we will show that for every m-tuple $(l_1, \ldots, l_m)$ simplicial complex K is a combinatorial sphere whenever $K(l_1, \ldots, l_m)$

is a combinatorial sphere. Then P is simple whenever $P(l_1,\dots,l_m)$ is simple (see example 2.1).

PROPOSITION 4.12 (Associativity law for the composition of simplicial complexes). *Let K be a simplicial complex on m vertices, $K_1,\dots,K_m$ be simplicial complexes on $l_1,\dots,l_m$ vertices respectively and $K_{11},\dots,K_{1l_1},K_{21},\dots,K_{2l_2},\dots,K_{m1},\dots,K_{ml_m}$ – simplicial complexes on sets $[r_{sj_s}]$ of vertices. Then*

$$K(K_1(K_{11},\dots,K_{1l_1}),\dots,K_m(K_{m1},\dots,K_{ml_m})) = \\ K(K_1,\dots,K_m)(K_{11},\dots,K_{1l_1},\dots,K_{m1},\dots,K_{ml_m}) \tag{4.1}$$

as the complexes on the set $\bigsqcup_{s,j_s}[r_{sj_s}]$.

PROOF. Both complexes have the same set of vertices $V = ([r_{11}]\sqcup\ldots\sqcup[r_{1l_1}])\sqcup\ldots\sqcup([r_{m1}]\sqcup\ldots\sqcup[r_{ml_m}])$ Let A be the subset of V so $A=(A_{11}\sqcup\ldots\sqcup A_{1l_1})\sqcup\ldots\sqcup(A_{m1}\sqcup\ldots\sqcup A_{ml_m})$, where $A_{sj_s}\subseteq[r_{sj_s}]$. The chain of equivalent conditions is written below.

$$\begin{aligned} A\in K(K_1(K_{11},\dots,K_{1l_1}),\dots,K_m(K_{m1},\dots,K_{ml_m})) &\Leftrightarrow \\ \exists I\in K\forall s\notin I\colon (A_{s1}\sqcup\ldots\sqcup A_{sl_s})\in K_s(K_{s1},\dots,K_{sl_s}) &\Leftrightarrow \\ \exists I\in K\forall s\notin I\exists I_s\in K_s\forall i_s\notin I_s\colon A_{si_s}\in K_{si_s} &\Leftrightarrow \\ \exists J\in K(K_1,\dots,K_m)\forall s\forall i_s\in[l_s]\setminus J\colon A_{si_s}\in K_{si_s} &\Leftrightarrow \\ A\in K(K_1,\dots,K_m)(K_{11},\dots,K_{1l_1},\dots,K_{m1},\dots,K_{ml_m}). \end{aligned} \tag{4.2}$$

This finishes the proof. □

REMARK 4.13. As in the case of polytopes simplicial complexes form an operad. The simplicial complex K on m vertices can be viewed as an m-adic operation. The "identity operation" is given by the complex o^1 (see example 4.6) since $K(o^1,\dots,o^1)=o^1(K)=K$.

COROLLARY 4.14. *The composition can be constructed by steps. More precisely, let K_i be the complex on $[l_i]$, then*

$$K(K_1,\dots,K_m) = K(o^1,\dots,K_i,\dots,o^1)(K_1,\dots,K_{i-1},\underbrace{o^1,\dots,o^1}_{l_i},K_{i+1},\dots,K_m).$$

COROLLARY 4.15 ([**5**, sect.2]). *Let l_i be natural numbers. Then*

$$K(l_1,\dots,l_m) = K(1,\dots,l_i,\dots,1)(l_1,\dots,l_{i-1},\underbrace{1,\dots,1}_{l_i},l_{i+1},\dots,l_m).$$

One can "blow up" vertices step by step. The operation $K(l_1,1,1,\dots,1)$ can be described geometrically [**5**], [**14**]:

$$K(l,1,1,\dots,1) = K_{[m]\setminus\{1\}} * \partial\Delta_{[l]} \cup (\mathrm{link}_K\{1\}) * \Delta_{[l]}.$$

The figure 3 illustrates the situation when K is the boundary of 5-gon and $l=2$.

It can be directly checked that $K(l,1,1,\dots,1)\cong_{PL}\Sigma^{l-1}K\cong_{PL}K*\partial\Delta_{[l]}$. In the case when K is the boundary of simplicial polytope the complex $K(l,1,1,\dots,1)$ is also the boundary of a polytope [**5**, Th.2.3]. Indeed, if $K=\partial Q$, then $K=K_{Q^*}$ for dual simple polytope Q^*, then $K(l_1,\dots,l_m)=K_{Q^*(l_1,\dots,l_m)}=\partial(Q^*(l_1,\dots,l_m))^*$. Then, using corollary 4.15 inductively, we get the following.

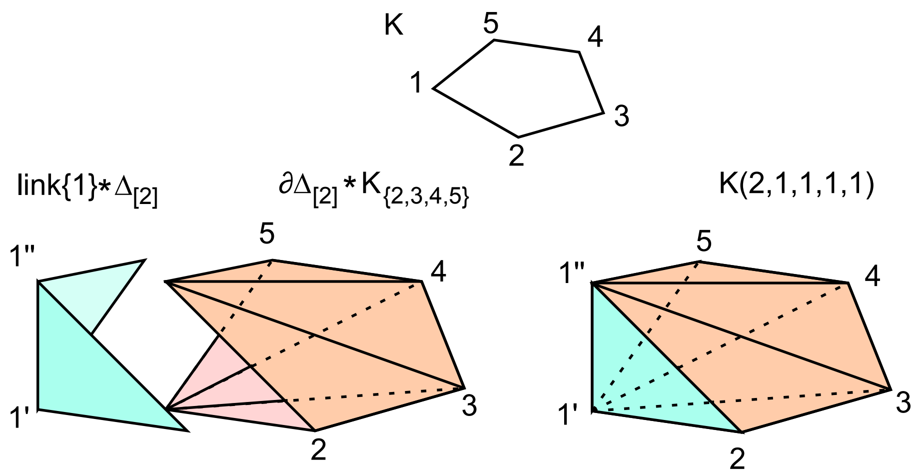

FIGURE 3. Doubling a vertex in a boundary of a pentagon

COROLLARY 4.16. *If K is a simplicial sphere (boundary of a simplicial polytope, triangulated topological sphere, homological sphere) then so is $K(l_1, \ldots, l_m)$ for each array $l_1, \ldots, l_m$ of natural numbers.*

There is the converse question: for which $K, K_1, \ldots, K_m$ the composed complex $K(K_1, \ldots, K_m)$ is a sphere (in any sense)? We postpone this question till section 6.

5. Polyhedral products given by composed complexes

In this section we describe the polyhedral products given by the composed simplicial complexes.

PROPOSITION 5.1. *Let K be a complex with m vertices and let $\{K_i\}_{i\in[m]}$ be simplicial complexes with l_i vertices. Consider topological pairs, indexed by the elements of the set $\bigsqcup_i [l_i]$:*

$$(X_{11}, A_{11}), \ldots, (X_{1l_1}, A_{1l_1}), \ldots, (X_{m1}, A_{m1}), \ldots, (X_{ml_m}, A_{ml_m}).$$

For each $i \in [m]$ let $Y_i = X_{i1} \times X_{i2} \times \ldots \times X_{il_i}$ and $Z_i = \mathcal{Z}_{K_i}(\underline{(X_{ij}, A_{ij})}_{j\in[l_i]}) \subseteq Y_i$. Then

$$\mathcal{Z}_{K(\underline{K_\alpha})}(\underline{(X_{ij}, A_{ij})}_{\substack{i\in[m]\\ j\in[l_i]}}) = \mathcal{Z}_K(\underline{(Y_i, Z_i)}_{i\in[m]})$$

as subsets of $\prod_{i,j} X_{ij} = \prod_i Y_i$.

The proof is similar to the proposition 4.12 and rather straightforward.

EXAMPLE 5.2. In the case $K_i = \partial\Delta_{[l_i]}$ the proposition 5.1 coincides with [**5**, Theorem 7.2].

EXAMPLE 5.3. Let $K, K_1, \ldots, K_m$ be as before. Consider the set of pairs $\{(X_{ij}, A_{ij})\}_{\substack{i\in[m],\\ j\in[l_i]}}$ where $(X_{ij}, A_{ij}) = \{(D^2, S^1)\}$ for each $i \in [m]$ and $j \in [l_i]$. Then

$$\mathcal{Z}_{K(\underline{K_\alpha})}(D^2, S^1) = \mathcal{Z}_K(\underline{(D^{2l_i}, \mathcal{Z}_{K_i})}_{i\in[m]}).$$

Example 5.4. As the particular case of the previous example consider $K_i = \partial\Delta_{[l_i]}$. Then

$$\mathcal{Z}_{K(l_1,\dots,l_m)}(D^2, S^1) = \mathcal{Z}_K(\underline{(D^{2l_i}, S^{2l_i-1})}_{i\in[m]})$$

since $\mathcal{Z}_{\partial\Delta_{[l_i]}}(D^2, S^1) = S^{2l_i-1}$ which coincides with [**5**, Corollary 7.3].

The similar statement holds for real moment-angle complexes

$$\mathcal{Z}_{K(l_1,\dots,l_m)}(D^1, S^0) = \mathcal{Z}_K(\underline{(D^{l_i}, S^{l_i-1})}_{i\in[m]})$$

In particular

$$\mathcal{Z}_{2K}(D^1, S^0) = \mathcal{Z}_{K(2,\dots,2)}(D^1, S^0) = \mathcal{Z}_K(D^2, S^1).$$

This idea was used by Yu.Ustinovsky in his work [**17**] to prove the toral rank conjecture for moment-angle spaces.

6. Combinatorial and topological properties of composed complexes

In [**4**] was proved that $\mathcal{Z}_K^{\wedge}(\underline{(X_i, A_i)}) \simeq \Sigma(K \wedge A_1 \wedge \ldots \wedge A_m)$ if $X_i \simeq \mathrm{pt}$ for each $i \in [m]$. The following statement can be proved by the similar argument.

Proposition 6.1. *Let (X_i, A_i) be topological pairs for $i \in [m]$ with X_i contractible. Then for every K on $[m]$ the space $\mathcal{Z}_K^*(\underline{(X_i, A_i)})$ is homotopy equivalent to $A_1 * \ldots * A_m * K$.*

Proof. Let $\mathrm{CAT}(K)$ be a small category associated to simplicial complex K. The objects of $\mathrm{CAT}(K)$ are the simplices of K and morphisms are inclusions.

Define a functor $\Psi\colon \mathrm{CAT}(K) \to \mathrm{TOP}$. For each $I \in \mathrm{Ob}\,\mathrm{CAT}(K)$ let $\Psi(I)$ be the space $U_I = B_1 * B_2 * \ldots * B_m$, where $B_i = X_i$ if $i \in I$ and $B_i = A_i$ otherwise. The morphism $\Psi(I \hookrightarrow I')$ is given by the natural inclusion $U_I \hookrightarrow U_{I'}$.

Then $\operatorname{colim}\Psi \cong \mathcal{Z}_K^*(\underline{(X_i, A_i)})$. All the maps in the diagram Ψ are closed cofibrations. Therefore, the projection lemma (see, e.g. [**18**, proposition 3.1]) implies $\operatorname{colim}\Psi \simeq \operatorname{hocolim}\Psi$.

Consider the diagram $\Phi\colon \mathrm{CAT}(K) \to \mathrm{TOP}$ given by $\Phi(\varnothing) = A_1 * \ldots * A_m$, $\Phi(I) = \mathrm{pt}$ if $I \neq \varnothing$. The values of Φ on the morphisms $I \subseteq I'$ are defined in the unique way.

1) For each $I \in \mathrm{Ob}\,\mathrm{CAT}(K)$ there is a homotopy equivalence $h_I\colon \Psi(I) \to \Phi(I)$. Indeed, for $I = \varnothing$ we have $\Psi(I) = A_1 * \ldots * A_m = \Phi(I)$ so $h_\varnothing$ can be chosen to be an identity map. For $I \neq \varnothing$ we have $\Psi(I) = B_1 * \ldots * B_m$ where at least one set B_i is equal to X_i thus contractible. Therefore the whole join $B_1 * \ldots * B_m$ is contractible. Then the unique map $h_I\colon \Psi(I) \to \Phi(I) = \mathrm{pt}$ is a homotopy equivalence.

2) The maps $h_I\colon \Psi(I) \to \Phi(I)$ are coherent, therefore, $\operatorname{hocolim}\Psi \simeq \operatorname{hocolim}\Phi$.

3) $\operatorname{hocolim}\Phi \simeq \Phi(\varnothing) * K \cong K * A_1 * \ldots * A_m$ (see [**18**, lemma 3.4]). This fact can be deduced from the constructive definition of a homotopy colimit.

4) The sequence of equivalences

$$\mathcal{Z}_K^*(\underline{(X_i, A_i)}) \cong \operatorname{colim}\Psi \simeq \operatorname{hocolim}\Psi \simeq \operatorname{hocolim}\Phi = K * A_1 * \ldots * A_m$$

completes the proof. □

Corollary 6.2. *For any simplicial complexes $K, K_1, \ldots, K_m$ with nonempty sets of vertices we have a homotopy equivalence*

$$K(\underline{K_\alpha}) \simeq K * K_1 * K_2 * \ldots * K_m.$$

Corollary 6.3. *If $K \simeq S^{n-1}$, $K_i \simeq S^{n_i-1}$ then $K(\underline{K_\alpha}) \simeq S^{n+n_1+\ldots+n_m-1}$.*

Corollary 6.4. *Let $P, P_1, \ldots, P_m$ be the polytopes of dimensions $n, n_1, \ldots, n_m$. Then $\dim P(\underline{P_\alpha}) = n + n_1 + \ldots + n_m$.*

PROOF. $K_Q \simeq S^{n-1}$ if $\dim Q = n$ for any convex polytope Q (see section 2). Therefore, $S^{\dim P(\underline{P_\alpha})-1} \simeq K_{P(\underline{P_\alpha})} = K_P(K_{P_1}, \dots, K_{P_m}) \simeq K_P * K_{P_1} * \dots * K_{P_m} \simeq S^{n+n_1+\dots+n_m-1}$. Then $\dim P(\underline{P_\alpha}) = n + n_1 + \dots + n_m$. $\square$

EXAMPLE 6.5. Let $K(o^{l_1}, \dots, o^{l_m})$ be the composition of K with "ghost complexes" o^l. Then $K(o^{l_1}, \dots, o^{l_m}) \simeq K$ by corollary 6.2. It can be seen from example 4.6 as well.

THEOREM 6.6. *Let $K(\underline{K_\alpha})$ be a simplicial sphere (resp. homological sphere) and suppose K does not have ghost vertices. Then K is a simplicial sphere (resp. homological sphere) and $K_i = \partial\Delta_{[l_i]}$ for some $l_i > 0$. If $K(\underline{K_\alpha})$ is the boundary of a simplicial polytope (up to ghost vertices), then so is K.*

PROOF. We need a technical lemma which describes the links of simplices in the composed complex $K(\underline{K_\alpha})$.

LEMMA 6.7. *Suppose $K, K_1, \dots, K_m$ are simplicial complexes on sets $[m], [l_1], \dots, [l_m]$. Let $A \in K(\underline{K_\alpha})$, $A = A_1 \sqcup \dots \sqcup A_m$, $A_i \subseteq [l_i]$ and $J = \{i \in [m] \mid A_i \notin K_i\} \in K$. Also let $\{i_1, \dots, i_k\} = [m] \setminus J$. For each $i \in J$ consider a set $M_i = [l_i] \setminus A_i$ and a simplex Δ_{M_i} spanned by this set. Then*

$$\operatorname{link}_{K(\underline{K_\alpha})} A = \operatorname{link}_K J(\operatorname{link}_{K_{i_1}} A_{i_1}, \dots, \operatorname{link}_{K_{i_k}} A_{i_k}) * (\mathop{\circledast}_{i\in J} \Delta_{M_i})\,.$$

PROOF. Both simplicial complexes have the same set of vertices $\bigsqcup_{i=1}^{m}([l_i] \setminus A_i)$. Let $I = I_1 \sqcup \dots \sqcup I_m \in \operatorname{link}_{K(\underline{K_\alpha})} A$. By definition this means $I \sqcup A \in K(\underline{K_\alpha})$. Equivalently,

$$B' = \{i \in [m] \mid I_i \sqcup A_i \notin K_i\} \in K.$$

Equivalently,

$$B = \{i \in [m] \setminus J \mid I_i \sqcup A_i \notin K_i\} \sqcup J \in K,$$

because $A_i \notin K_i$ yields $A_i \sqcup I_i \notin K_i$ and, therefore, $J \subseteq B'$. Equivalently,

$$B = \{i \in [m] \setminus J \mid I_i \sqcup A_i \notin K_i\} \in \operatorname{link}_K J.$$

Equivalently,

$$B = \{i \in [m] \setminus J \mid I_i \notin \operatorname{link}_{K_i} A_i\} \in \operatorname{link}_K J. \tag{6.1}$$

So far

$$I = \bigsqcup_{i\notin J} I_i \sqcup \bigsqcup_{i\in J} I_i,$$

where $\bigsqcup_{i\notin J} I_i$ satisfies (6.1), therefore

$$\bigsqcup_{i\notin J} I_i \in \operatorname{link}_K J(\operatorname{link}_{K_{i_1}} A_{i_1}, \dots, \operatorname{link}_{K_{i_k}} A_{i_k}).$$

Since no conditions on $\bigsqcup_{i\in J} I_i$ are imposed, we get the required formula. $\square$

Now let $K(\underline{K_\alpha})$ be a simplicial (resp. homological) sphere. Then for any $A \in K(\underline{K_\alpha})$ the complex $\operatorname{link}_{K(\underline{K_\alpha})} A$ is a simplicial (resp. homological) sphere as well. First of all, note that $K_i \neq \Delta_{[l_i]}$. Indeed, otherwise $K(\underline{K_\alpha}) \simeq K * K_1 * \dots * K_m$ is contractible which contradicts the assumption.

In what follows we use the notation of lemma 6.7. Suppose there exists a number $j \in [m]$ for which K_j has a nonsimplex $A_j \notin K_j$ such that $A_j \neq [l_j]$. Consider the subset

$$A = \varnothing \sqcup \ldots \sqcup A_j \sqcup \ldots \sqcup \varnothing \subseteq [l_1] \sqcup \ldots \sqcup [l_m].$$

Since $J = \{i \in [m] \mid A_i \notin K_i\} = \{j\} \in K$ by the assumption, we have $A \in K(\underline{K_\alpha})$. By lemma 6.7 $\operatorname{link}_{K(\underline{K_\alpha})} A = X * \Delta_{M_j}$, where X is some complex and Δ_{M_j} is a simplex spanned by $[l_j] \setminus A_j \neq \varnothing$. Therefore $\operatorname{link}_{K(\underline{K_\alpha})} A$ is contractible which contradicts the assumption that it is a sphere.

Thus for each i the only nonsimplices of K_i are $[l_i]$. This argument shows that $K_i = \partial\Delta_{[l_i]}$.

Let $I_i^{max} \in K_i = \partial\Delta_{[l_i]}$ be any maximal simplex (facet) for each $i \in [m]$. Then $l_i - |I_i^{max}| = 1$ and $\operatorname{link}_{K_i} I_i^{max} = o^1$, the complex on one ghost vertex.

Consider the simplex $A = I_1^{max} \sqcup \ldots \sqcup I_m^{max} \in K(\underline{K_\alpha})$. Since $J = \{i \in [m] \mid I_i \notin K_i\} = \varnothing$ we have $\operatorname{link}_K J = \operatorname{link}_K \varnothing = K$. Applying lemma 6.7 to the simplex A we get

$$\operatorname{link}_{K(\underline{K_\alpha})} A = \operatorname{link}_K J(\operatorname{link}_{K_1} I_1^{max}, \ldots, \operatorname{link}_{K_m} I_m^{max}) = K(o, \ldots, o) = K,$$

Since $\operatorname{link}_{K(\underline{K_\alpha})} A$ is a combinatorial (resp. homological) sphere, so is K.

If $K(\underline{K_\alpha})$ is the boundary of a simplicial polytope then all its links are the boundaries of simplicial polytopes. This gives the last part of the proposition. □

Remark 6.8. More general result can be obtained by the same arguments. Let $K(\underline{K_\alpha})$ be a combinatorial (homological) sphere. Then

1) K is a combinatorial sphere;
2) If i is not a ghost vertex of K, then $K_i = \partial\Delta_{[l_i]}$ for some $l_i > 0$.
3) If i is a ghost vertex of K, then K_i is a combinatorial (homological) sphere.

Proposition 6.9. *Let $K(\underline{K_\alpha}) = K_Q$ for some simple polytope Q. Then there exists a simple polytope P and numbers $l_i > 0$ such that $K = K_P$ and $K_i = \partial\Delta_{[l_i]}$. The polytopes Q and $P(l_1, \ldots, l_m)$ are combinatorially equivalent.*

Proof. If Q is simple $K_Q = \partial Q^*$. Therefore, if $K(\underline{K_\alpha}) = K_Q$, then $K(\underline{K_\alpha})$ is the boundary of a simplicial polytope and by theorem 6.6 $K_i = \partial\Delta_{[l_1]}$ and K is the boundary of a simplicial polytope. Then $K = K_P$ and $K_Q = K(\underline{K_\alpha}) = K_P(\partial\Delta_{[l_1]}, \ldots, \partial\Delta_{[l_m]}) = K_{P(l_1,\ldots,l_m)}$ by proposition 4.8. This means that Q and $P(l_1, \ldots, l_m)$ are combinatorially equivalent (see section 2). □

Proposition 4.8 motivated the assumption, that the class of spherical nerve-complexes is closed under composition unlike the class of simplicial spheres.

Theorem 6.10. *Let K be the spherical nerve-complex of rank n with m vertices and $K_1, \ldots, K_m$ be spherical nerve-complexes on $[l_1], \ldots, [l_m]$ of ranks $n_1, \ldots, n_m$ respectively. Then $K(K_1, \ldots, K_m)$ is a spherical nerve-complex of rank $n + n_1 + \ldots + n_m$.*

Proof. We use the notation of section 2. Let us describe the set of maximal simplices $M(K(\underline{K_\alpha}))$ and the set of their intersections $F(K(\underline{K_\alpha}))$. We have $I_1 \sqcup \ldots \sqcup I_m \in M(K(\underline{K_\alpha}))$ iff there exist a simplex $I \in M(K)$, such that $I_j = [l_j]$ for $j \in I$ and $I_j \in M(K_j)$ for $j \notin I$. Then $I_1 \sqcup \ldots \sqcup I_m \in F(K(\underline{K_\alpha}))$ iff there exists $I \in F(K)$, such that $I_j = [l_j]$ for $j \in I$ and $I_j \in F(K_j)$ for $j \notin I$. In this case we will say that I is the support of $I_1 \sqcup \ldots \sqcup I_m$. Obviously $\varnothing \in F(K(\underline{K_\alpha}))$.

The poset $F(K(\underline{K_\alpha}))$ is graded by the rank function

$$\operatorname{rank}_{F(K(\underline{K_\alpha}))}(I_1 \sqcup \ldots \sqcup I_m) = \operatorname{rank}'_{F(K_1)} I_1 + \ldots + \operatorname{rank}'_{F(K_m)} I_m + \operatorname{rank}_{F(K)} I,$$

where $\operatorname{rank}'_{F(K_j)} I_j = \operatorname{rank}_{F(K_j)} I_j$ if $I_j \in F(K_j)$ (that is $j \notin I$) and $\operatorname{rank}'_{F(K_j)} I_j = n_j$, the rank of the nerve-complex K_j, if $I_j = V_j$ (in the case $j \in I$).

For a link of $I_1 \sqcup \ldots \sqcup I_m$ with the support I we have

$$\operatorname{link}_{K(\underline{K_\alpha})}(I_1 \sqcup \ldots \sqcup I_m) = \operatorname{link}_K I \left(\underline{\{\operatorname{link}_{K_j} I_j\}}_{j \notin I}\right).$$

By corollary 6.2

$$\text{(6.2)} \quad \operatorname{link}_K I \left(\underline{\{\operatorname{link}_{K_j} I_j\}}_{j\notin I}\right) \simeq \operatorname{link}_K I * \left(\circledast_{j\notin I} \operatorname{link}_{K_j} I_j\right) \simeq$$
$$\simeq S^{n-\operatorname{rank}_{F(K)} I - 1} * \left(\circledast_{j\notin I} S^{n_j - \operatorname{rank}_{F(K_j)} I_j - 1}\right) \cong S^{n + \sum\limits_{j\notin I} n_j - \operatorname{rank}_{F(K)} I - \sum\limits_{j \notin I} \operatorname{rank}_{F(K_j)} I_j - 1}.$$

Since $\operatorname{rank}'_{F(K_j)} I_j = n_j$ in the case $j \in I$, by adding and subtracting $\sum_{j\in I} \operatorname{rank}'_{F(K_j)} I_j$ to the dimension of a sphere in the last expression we get

$$\text{(6.3)} \quad n + \sum_{j\notin I} n_j - \operatorname{rank}_{F(K)} I - \sum_{j\notin I} \operatorname{rank}_{F(K_j)} I_j - 1 =$$
$$= n + \sum_{j\in[m]} n_j - \sum_{j\in[m]} \operatorname{rank}'_{F(K_j)} I_j - 1 = n + \sum_{j\in[m]} n_j - \operatorname{rank}_{F(K(\underline{K_\alpha}))}(I_1 \sqcup \ldots \sqcup I_m) - 1,$$

and the statement follows. □

7. Multigraded Betti numbers of the compositions

In this section we review the definition of multigraded Betti numbers of a simplicial complex K, and the Hochster formula expressing multigraded Betti numbers as the ranks of cohomology groups of full subcomplexes in K. Together with corollary 6.2 Hochster formula will give an explicit formula expressing multigraded Betti numbers of $K(\underline{K_\alpha})$ in terms of multigraded Betti numbers of $K, K_1, \ldots, K_m$. In particular cases this formula has very simple form and allows to find the h-vectors of composed complexes.

Let $\Bbbk$ be the ground field and $\Bbbk[m] = \Bbbk[v_1, \ldots, v_m]$ — the ring of polynomials in m indeterminates. The ring $k[m]$ has a $\mathbb{Z}^m$-grading defined by $\deg v_i = (0, \ldots, 2, \ldots, 0)$ with 2 on the i-th place. The field $\Bbbk$ is given the $\Bbbk[m]$-module structure via the epimorphism $\Bbbk[m] \to \Bbbk$, $v_i \mapsto 0$.

Let K be a simplicial complex on m vertices. The Stanley–Reisner algebra $\Bbbk[K]$ is defined as a quotient algebra $\Bbbk[m]/I_{SR}$, where the Stanley–Reisner ideal I_{SR} is generated by square-free monomials $v_{\alpha_1} \ldots v_{\alpha_k}$ corresponding to nonsimplices $\{\alpha_1, \ldots, \alpha_m\} \notin K$. The algebra $\Bbbk[K]$ has a natural $\Bbbk[m]$-module structure, given by quotient epimorphism $\Bbbk[m] \to \Bbbk[m]/I_{SR}$. Since I_{SR} is homogeneous ideal, the module $\Bbbk[m]$ is $\mathbb{Z}^m$-graded.

Let $\ldots \to R^{-i} \to R^{-i+1} \to \ldots \to R^{-1} \to R^0 \to \Bbbk[K]$ be a free resolution of the module $\Bbbk[K]$ by $\mathbb{Z}^m$-graded $\Bbbk[m]$-modules R^{-i}. We have $R^{-i} = \bigoplus_{\bar{j}\in\mathbb{Z}^m} R^{-i,\bar{j}}$. The Tor-module of a complex K therefore has a natural $\mathbb{Z}^{m+1}$-grading:

$$\operatorname{Tor}_{\Bbbk[m]}(\Bbbk[K], \Bbbk) = \bigoplus_{i\in\mathbb{Z}_{\geqslant};\bar{j}\in\mathbb{Z}^m} \operatorname{Tor}^{-i,2\bar{j}}_{\Bbbk[m]}(\Bbbk[K], \Bbbk).$$

The multigraded Betti numbers of a complex K are defined as the dimensions of the graded components of the Tor-module:

$$\beta_{\Bbbk}^{-i,2\bar{j}}(K) = \dim_{\Bbbk} \mathrm{Tor}_{\Bbbk[m]}^{-i,2\bar{j}}(\Bbbk[K], \Bbbk).$$

These numbers depend on the ground field but we will omit $\Bbbk$ to avoid cumbersome notation. A combinatorial description of multigraded Betti numbers is given by Hochster formula [**11, 7**].

THEOREM 7.1 (Hochster, [**9**, Th.3.2.8]). *For a simplicial complex K on m vertices and $\bar{j} = (j_1, \ldots, j_m) \in \mathbb{Z}^m$ there holds $\beta^{-i,2\bar{j}} = 0$ if $\bar{j} \notin \{0,1\}^m$. If $\bar{j} \in \{0,1\}^m$ and $A = \{i \in [m] \mid j_i = 1\}$, then*

$$\beta^{-i,2\bar{j}} = \tilde{H}^{|A|-i-1}(K_A; \Bbbk), \tag{7.1}$$

where K_A is a full subcomplex of K on the set of vertices A. In this formula it is assumed that $\tilde{H}^{-1}(\varnothing; \Bbbk) = \Bbbk$.

By this result the set of all multigraded Betti numbers is a complete combinatorial invariant of a simplicial complex.

If $A \subseteq [m]$ we use the notation $\beta^{-i,2A}$ for the number $\beta^{-i,2\bar{j}}$, where $\bar{j} = (j_1, \ldots, j_m)$, $j_i = 1$ if $i \in A$ and $j_i = 0$ otherwise.

REMARK 7.2. The full subcomplex will be sometimes denoted by $K|_A$ instead of K_A, especially in the case when for K stands a complex with its own lower index.

Bigraded Betti numbers are defined by the formula $\beta^{-i,2j}(K) = \sum_{|A|=j} \beta^{-i,2A}$. These numbers are the dimensions of graded components of the Tor-algebra $\mathrm{Tor}_{\Bbbk[m]}^{*,*}(\Bbbk[K], \Bbbk)$ if we specialize the $\mathbb{Z}^m$-grading $(j_1, \ldots, j_m)$ to the $\mathbb{Z}$-grading $\sum j_i$.

To work with multigraded Betti numbers we construct their generating functions, called beta-polynomials of K. Let

$$\beta_K(s, \bar{t}) = \beta_K(s, t_1, t_2, \ldots, t_m) = \sum_{i \in \mathbb{Z}, \bar{j} \in \mathbb{Z}^m} \beta^{-i,2\bar{j}}(K) s^i \bar{t}^{\bar{j}},$$

where $\bar{t}^{\bar{j}}$ stands for the monomial $t_1^{j_1} t_2^{j_2} \ldots t_m^{j_m}$.

By Hochster formula

$$\beta_K(s, \bar{t}) = \sum_{i \in \mathbb{Z}, A \subseteq [m]} \beta^{-i,2A}(K) s^i \bar{t}^A,$$

where $\bar{t}^A = \prod_{l \in A} t_l$. The free term, which corresponds to $A = \varnothing, i = 0$, equals 1 for any complex K. In what follows we need the reduced beta-polynomial

$$\tilde{\beta}_K(s, t) = \beta_K(s, t) - 1 = \sum_{\substack{i \in \mathbb{Z} \\ A \subseteq [m], A \neq \varnothing}} \beta^{-i,2A} s^i \bar{t}^A.$$

Two-parametric beta-polynomial (see [**3**, sect.8]) is defined as

$$b_K(s, t) = \sum_{i,j \in \mathbb{Z}} \beta^{-i,2j} s^{-i} t^{2j} = \beta_K(s^{-1}, t^2, t^2, \ldots, t^2).$$

and

$$\tilde{b}_K(s, t) = \sum_{i,j \in \mathbb{Z}, j \neq 0} \beta^{-i,2j} s^{-i} t^{2j} = \tilde{\beta}_K(s^{-1}, t^2, t^2, \ldots, t^2) = b_K(s, t) - 1.$$

EXAMPLE 7.3. Let $K = \partial\Delta_{[m]}$. Then by Hochster formula we have

$$\beta_{\partial\Delta_{[m]}}(s,\bar{t}) = 1 + st_1t_2\dots t_m$$

since the nonacyclic full subcomplexes of K are only $K_\varnothing$ and $K_{[m]} = K$. These subcomplexes have nontrivial reduced cohomology in dimensions -1 and $m-2$ respectively.

EXAMPLE 7.4. Let $K = o^m$. Then

$$\beta_{o^m}(s,\bar{t}) = \sum_{A\subseteq[m]} s^{|A|}\bar{t}^A,$$

since for any $A \subseteq [m]$ the full subcomplex $(o^m)_A$ is empty and its (-1)-cohomology has rank 1. Therefore,

$$\beta_{o^m}(s,\bar{t}) = (1+st_1)\cdot\ldots\cdot(1+st_m). \tag{7.2}$$

For the polytope P we define the polynomials $\beta, \tilde{\beta}, b, \tilde{b}$ as the polynomials of the corresponding nerve-complex K_P:

$$\beta_P(s,\bar{t}) = \beta_{K_P}(s,\bar{t}), \qquad \tilde{\beta}_P(s,\bar{t}) = \tilde{\beta}_{K_P}(s,\bar{t}), \tag{7.3}$$

$$b_P(s,t) = b_{K_P}(s,t), \qquad \tilde{b}_P(s,t) = \tilde{b}_{K_P}(s,t). \tag{7.4}$$

Now we generalize some results of **[3]** concerning beta-polynomials. Our goal is to express $\beta_{K(\underline{K_\alpha})}(s,\bar{t})$ in terms of $\beta_K(s,\bar{t})$ and $\beta_{K_i}(s,\bar{t})$. To do this at first we investigate the structure of full subcomplexes in $K(\underline{K_\alpha})$.

LEMMA 7.5. *Consider K on m vertices and K_α on l_α vertices for $\alpha \in [m]$, so the set of vertices of $K(\underline{K_\alpha})$ is $[l_1]\sqcup\ldots\sqcup[l_m]$. Let A be the subset of $[l_1]\sqcup\ldots\sqcup[l_m]$, $A = A_1\sqcup\ldots\sqcup A_m$ where $A_\alpha \subseteq [l_\alpha]$. Let $\nu = \{\alpha_1,\ldots,\alpha_k\} = \{\alpha\in[m] \mid A_\alpha \neq \varnothing\}$. Then*

$$K(K_1,\ldots,K_m)_A = K|_\nu(K_{\alpha_1}|_{A_{\alpha_1}}, K_{\alpha_2}|_{A_{\alpha_2}},\ldots,K_{\alpha_k}|_{A_{\alpha_k}}).$$

The proof follows from definitions.

THEOREM 7.6. *Let K be the complex on m vertices and $K_1,\ldots,K_m$ be simplicial complexes on $l_1,\ldots,l_m$ vertices. Let $\bar{t}_j = (t_{j1},\ldots,t_{jl_j})$ for $j\in[m]$ and*

$$\bar{t} = (t_{11},\ldots,t_{1l_1},\ldots,t_{m1},\ldots,t_{ml_m}) = (\bar{t}_1,\ldots,\bar{t}_m).$$

Then

$$\beta_{K(K_1,\ldots,K_m)}(s,\bar{t}) = \beta_K(s, s^{-1}\tilde{\beta}_{K_1}(s,\bar{t}_1), s^{-1}\tilde{\beta}_{K_2}(s,\bar{t}_2),\ldots,s^{-1}\tilde{\beta}_{K_m}(s,\bar{t}_m)). \tag{7.5}$$

PROOF. Using Hochster formula 7.1 we may write

$$\beta_{K(\underline{K_\alpha})}(s,\bar{t}) = \sum_{A\subseteq[l_1]\sqcup\ldots\sqcup[l_m]} \sum_{i'} \dim \tilde{H}^{|A|-i'-1}(K(\underline{K_\alpha})_A;\Bbbk)s^{i'}\bar{t}^A = \sum_{A\subseteq[l_1]\sqcup\ldots\sqcup[l_m]} \sum_{i'} H_{i',A}s^{i'}\bar{t}^A, \tag{7.6}$$

where $H_{i',A}$ denote the dimensions of cohomology groups. Any subset $A \in [l_1]\sqcup\ldots\sqcup[l_m]$ is given by $A = A_1\sqcup\ldots\sqcup A_k$ for some $B = \{\alpha_1,\ldots,\alpha_k\}\subseteq[m]$ and $A_1\subseteq[l_{\alpha_1}],\ldots,A_k\subseteq[l_{\alpha_k}]$ subject to the condition $A_i\neq\varnothing$. The sum in (7.6) can be expanded

$$\sum_{A\subseteq[l_1]\sqcup\ldots\sqcup[l_m]}\sum_{i'}H_{i',A}s^{i'}\bar{t}^A = \sum_{i'}\sum_{B=\{\alpha_1,\ldots,\alpha_k\}\subseteq[m]}\left(\sum_{\substack{A_1\subseteq[l_{\alpha_1}]\\A_1\neq\varnothing}}\cdots\sum_{\substack{A_k\subseteq[l_{\alpha_k}]\\A_1\neq\varnothing}}H_{i',A}s^{i'}\bar{t}_{\alpha_1}^{A_1}\ldots\bar{t}_{\alpha_k}^{A_k}\right). \tag{7.7}$$

Quantities $H_{i',A}$ can be expressed using lemma 7.5 and corollary 6.2:

$$\begin{aligned} H_{i',A} &= \dim\tilde{H}^{|A|-i'-1}(K(\underline{K_\alpha})_A;\Bbbk) = \\ &\dim\tilde{H}^{|A_1|+\ldots+|A_k|-i'-1}(K|_B(K_{\alpha_1}|_{A_1},\ldots,K_{\alpha_k}|_{A_k});\Bbbk) = \\ &\dim\tilde{H}^{|A_1|+\ldots+|A_k|-i'-1}(K|_B*K_{\alpha_1}|_{A_1}*\ldots*K_{\alpha_k}|_{A_k};\Bbbk). \end{aligned} \tag{7.8}$$

The cohomology group of the join can be expanded

$$\dim\tilde{H}^{|A_1|+\ldots+|A_k|-i'-1}(K|_B*K_{\alpha_1}|_{A_1}*\ldots*K_{\alpha_k}|_{A_k};\Bbbk) = \sum_{\substack{r,r_1,\ldots,r_k\\r+r_1+\ldots+r_k=|A_1|+\ldots+|A_k|-i'-1-k}}\dim\tilde{H}^{r}(K|_B;\Bbbk)\cdot\dim\tilde{H}^{r_1}(K_{\alpha_1}|_{A_1};\Bbbk)\cdot\ldots\cdot\dim\tilde{H}^{r_k}(K_{\alpha_k}|_{A_k};\Bbbk). \tag{7.9}$$

Consider indices $i,i_1,\ldots,i_k$ satisfying the identities $r=k-i-1=|B|-i-1$, $r_s=|A_s|-i_s-1$ for $s\in[k]$. Since $r+\sum_{s\in[k]}r_s=\left(\sum_{s\in[k]}|A_s|\right)-i'-1-k$, we get $i'=i-k+\sum_{s\in[k]}i_s$. Then

$$\sum_{r,r_1,\ldots,r_k}\dim\tilde{H}^{r}(K|_B;\Bbbk)\cdot\left(\prod_{j=1}^{k}\dim\tilde{H}^{r_j}(K_{\alpha_j}|_{A_j};\Bbbk)\right)s^{i'}\bar{t}_{\alpha_1}^{A_1}\ldots\bar{t}_{\alpha_k}^{A_k} = \sum_{i_1,\ldots,i_k}\dim\tilde{H}^{k-i-1}(K|_B;\Bbbk)s^{i}\prod_{j=1}^{k}\left(s^{-1}\dim\tilde{H}^{|A_j|-i_j-1}(K_{\alpha_j}|_{A_j};\Bbbk)s^{i_j}\bar{t}_{\alpha_j}^{A_j}\right). \tag{7.10}$$

Therefore,

$$\sum_{\substack{A_1\subseteq[l_{\alpha_1}]\\ A_1\neq\varnothing}} \cdots \sum_{\substack{A_k\subseteq[l_{\alpha_k}]\\ A_k\neq\varnothing}} \sum_{i_1,\dots,i_k} \prod_{j=1}^{k} \left(s^{-1}\dim \tilde{H}^{|A_j|-i_j-1}(K_{\alpha_j}|_{A_j};\Bbbk)s^{i_j}\bar{t}_{\alpha_j}^{A_j}\right) = \tag{7.11}$$

$$\prod_{j=1}^{k}\left(\sum_{\substack{A_j\subseteq[l_{\alpha_j}]\\ A_j\neq\varnothing}} \sum_{i_j} s^{-1}\dim \tilde{H}^{|A_j|-i_j-1}(K_{\alpha_j}|_{A_j};\Bbbk)s^{i_j}\bar{t}_{\alpha_j}^{A_j}\right) =$$

$$\prod_{j=1}^{k}\left(\sum_{\substack{A_j\subseteq[l_{\alpha_j}]\\ A_j\neq\varnothing}} \sum_{i_j} s^{-1}\beta^{-i_j,2A_j}(K_{\alpha_j})s^{i_j}\bar{t}_{\alpha_j}^{A_j}\right) = \prod_{j=1}^{k}\left(s^{-1}\tilde{\beta}_{K_{\alpha_j}}(s,\bar{t}_{\alpha_j})\right).$$

Substituting (7.11) into (7.7) we get

$$\beta_{K(\underline{K_\alpha})}(s,\bar{t}) = \tag{7.12}$$

$$\sum_{i}\sum_{B=\{\alpha_1,\cdots,\alpha_k\}\subseteq[m]} \beta^{-i,2B}(K|_B)s^i\prod_{j=1}^{k}\left(s^{-1}\tilde{\beta}_{K_{\alpha_j}}(s,\bar{t}_{\alpha_j})\right) =$$

$$\beta_K(s,s^{-1}\tilde{\beta}_{K_1}(s,\bar{t}_1),\dots,s^{-1}\tilde{\beta}_{K_m}(s,\bar{t}_m)).$$

This finishes the proof. □

Corollary 7.7.

$$b_{K(\underline{K_\alpha})}(s,t) = \beta(s^{-1},s\tilde{b}_{K_1}(s,t),\dots,s\tilde{b}_{K_m}(s,t)).$$

Proof. Substitute s^{-1} and t^2 for s and t_{ji_j} in (7.5) and use the definition of a two-parametric beta-polynomial. □

Corollary 7.8. *Let P_1 and P_2 be two convex polytopes and $P_1 * P_2$ — their join. Let $\bar{t}_i = (t_{i1},\dots,t_{il_i})$ be formal variables corresponding to facets of P_i for $i = 1,2$ and $\bar{t} = (\bar{t}_1,\bar{t}_2)$. Then*

$$\beta_{P_1*P_2}(s,\bar{t}) = 1 + s^{-1}\tilde{\beta}_{P_1}(s,\bar{t}_1)\tilde{\beta}_{P_2}(s,\bar{t}_2)$$
$$b_{P_1*P_2}(s,t) = 1 + s\tilde{b}_{P_1}(s,t)\tilde{b}_{P_2}(s,t)$$

Proof. By example 3.2 we have $P_1*P_2 = \triangle_{[2]}(P_1,P_2)$. By proposition 4.8 $K_{\triangle_{[2]}(P_1,P_2)} = K_{\triangle_{[2]}}(K_{P_1},K_{P_2}) = \partial\Delta_{[2]}(K_{P_1},K_{P_2})$. Then by definition

$$\beta_{P_1*P_2}(s,\bar{t}) = \beta_{\triangle_{[2]}(P_1,P_2)}(s,\bar{t}) = \beta_{\partial\Delta_{[2]}}(s,s^{-1}\tilde{\beta}_{P_1}(s,\bar{t}_1),s^{-1},\tilde{\beta}_{P_2}(s,\bar{t}_2)) = \tag{7.13}$$
$$1 + s^{-1}\tilde{\beta}_{P_1}(s,\bar{t}_1)\tilde{\beta}_{P_2}(s,\bar{t}_2).$$

Substituting s^{-1} for s and t^2 for each t_{rj} gives the second expression of the corollary. See **[3]** for an independent proof of this statement. □

Corollary 7.9. *Let K be a simplicial complex on m vertices and $(l_1,\dots,l_m)$ — an array of nonnegative integers. Then*

$$b_{K(l_1,\dots,l_m)}(s,t) = \beta_K(s^{-1},t^{2l_1},t^{2l_2},\dots,t^{2l_m})$$

In particular, if $l_1 = l_2 = \ldots = l_m = l$ *we have*

$$b_{K(\underline{l})}(s,t) = b_K(s,t^l).$$

Proof. By definition $K(l_1,\ldots,l_m) = K(\partial\Delta_{[l_1]},\ldots,\partial\Delta_{[l_m]})$ and $\tilde{\beta}_{\partial\Delta_{[l_r]}}(s,\bar{t}_r) = st_{r1}\ldots t_{rl_r}$ (example 7.3). Then

$$\beta_{K(l_1,\ldots,l_m)}(s,\bar{t}) = \beta_K(s, s^{-1}st_{11}\ldots t_{1l_1},\ldots,s^{-1}st_{m1}\ldots t_{ml_m}) = \\ \beta_K(s, t_{11}\cdot\ldots\cdot t_{1l_1},\ldots,t_{m1}\cdot\ldots\cdot t_{ml_m}).$$

Substituting s^{-1} for s and t^2 for t_{rj} gives the required formula. $\square$

Example 7.10. Consider the case $o^m(K_1,\ldots,K_m) = K_1 * \ldots * K_m$. Using theorem 7.6 and relation (7.2) we get

$$\beta_{K_1*\ldots*K_m}(s,\bar{t}) = \beta_{K_1*\ldots*K_m}(s,\bar{t}) = \\ (1+s\cdot s^{-1}\tilde{\beta}_{K_1}(s,\bar{t}_1))\cdot\ldots\cdot(1+s\cdot s^{-1}\tilde{\beta}_{K_m}(s,\bar{t}_m)) = \beta_{K_1}(s,\bar{t}_1)\cdot\ldots\cdot\beta_{K_m}(s,\bar{t}_m). \tag{7.14}$$

This result can be proved directly by the isomorphism

$$\Bbbk[K_1 * \ldots * K_m] \cong \Bbbk[K_1]\otimes\ldots\otimes\Bbbk[K_m].$$

and the definition of multigraded Betti numbers.

8. Enumerative polynomials

Let K be a simplicial complex. For each $i \geqslant 0$ define a number $f_i = |\{I \in K \mid |I| = i\}|$. The polynomial

$$f_K(t) = \sum_i f_i t^i = \sum_{I\in K} t^{|I|}$$

is called an f-polynomial of K. If $\dim K = n-1$, then $\deg f_K(t) = n$. The h-numbers h_i are defined by the relation

$$h_0t^n + \ldots + h_{n-1}t + h_n = f_0(t-1)^n + f_1(t-1)^{n-1} + \ldots + f_n.$$

The polynomial $h_K(t) = h_0 + h_1t + \ldots + h_nt^n$ is called the h-polynomial of the complex K. Writing the defining relations for h_i we have

$$h_K(t) = (1-t)^n f_K\left(\frac{t}{1-t}\right). \tag{8.1}$$

Since the relation (8.1) is invertible, h-and f-polynomials carry the same combinatorial information. The h-polynomial is connected to Hilbert–Poincare series of the algebra $\Bbbk[K]$ with $\mathbb{Z}$-grading by the formula **[15]**,**[7]**:

$$\mathrm{Hilb}(\Bbbk[K];t) = \frac{h_K(t^2)}{(1-t^2)^n}. \tag{8.2}$$

There is a formula which connects h-polynomial of K with bigraded Betti numbers. Let $\chi_j(K) = \sum_{i=0}^m (-1)^i\beta^{-i,2j}(K)$ and $\chi_K(t) = \sum_{j=0}^m \chi_j(K)t^{2j}$. Then by [**7**, Theorem 7.15]

$$\chi_K(t) = (1-t^2)^{m-n}h_K(t^2) = (1-t^2)^m\,\mathrm{Hilb}(\Bbbk[K];t). \tag{8.3}$$

Since $\chi_K(t) = b_K(-1,t)$ we get a simple formula

$$b(-1,t) = (1-t^2)^{m-n}h_K(t^2). \tag{8.4}$$

Equation 8.4 allows to express the h-polynomial of $lK = K(l,\dots,l)$ in terms of the h-polynomial of K.

PROPOSITION 8.1. *Let K be $(n-1)$-dimensional complex on m vertices, $l > 0$ and $lK = K(l,l,\dots,l) = K(\partial\Delta_{[l]},\dots,\partial\Delta_{[l]})$. Then*

$$h_{lK}(t) = (1+t+\ldots+t^{l-1})^{m-n}h_K(t^l)$$

PROOF. The complex lK has $m' = ml$ vertices. It can be seen that $n' = \dim lK + 1 = nl + (m-n)(l-1)$. Then $m'-n' = m-n$. By relation (8.4) $b_{lK}(-1,t) = (1-t^2)^{m'-n'}h_{lK}(t^2)$. On the other hand, by corollary 7.9 $b_{lK}(s,t) = b_K(s,t^l)$, therefore $b_{lK}(-1,t) = b_K(-1,t^l)$. This gives a sequence of equalities:

$$(1-t^2)^{m-n}h_{lK}(t^2) = (1-t^2)^{m'-n'}h_{lK}(t^2) = b_{lK}(-1,t) = b_K(-1,t^l) = (1-t^{2l})^{m-n}h_K(t^{2l}).$$

Therefore, $h_{lK}(t^2) = \left(\frac{1-t^{2l}}{1-t^2}\right)^{m-n} h_K(t^{2l}) = (1+t^2+\ldots+t^{2(l-1)})^{m-n}h_K(t^{2l})$. □

In particular for $l = 2$ this gives $h_{2K}(t) = (1+t)^{m-n}h_K(t^2)$. This result is proved in **[16]** by another method.

REMARK 8.2. The result of proposition 8.1 can be proved independently using formula (8.2) by studying the structure of the ring $\Bbbk[lK]$ (see **[5]** for details).

It is convenient to introduce another polynomial $q_K(t)$ while working with the composition of simplicial complexes. For an $(n-1)$-dimensional complex K with m vertices let

$$q_K(t) = 1-(1-t)^{m-n}h_K(t).$$

EXAMPLE 8.3. It is known that $h_{\partial\Delta_{[m]}}(t) = 1+t+\ldots+t^{m-1}$. Then $q_{\partial\Delta_{[m]}}(t) = t^m$.

We have a formula

$$\tilde{b}_K(-1,t) = b_K(-1,t) - 1 = (1-t^2)^{m-n}h_K(t) - 1 = -q_K(t^2). \tag{8.5}$$

Also we have

$$\tilde{\beta}_K(-1,t,\dots,t) = -q_K(t) \tag{8.6}$$

PROPOSITION 8.4. *Consider arbitrary simplicial complexes $K_1,\dots,K_m$. Then*

$$q_{\partial\Delta_{[m]}(K_1,\dots,K_m)} = q_{K_1}(t)\cdot\ldots\cdot q_{K_m}(t)$$

PROOF. By corollary 7.7 $\tilde{b}_{\partial\Delta_{[m]}(\underline{K_\alpha})}(s,t) = \tilde{\beta}_{\partial\Delta_{[m]}}(s^{-1}, s\tilde{b}_{K_1}(s,t),\dots,s\tilde{b}_{K_m}(s,t)) = s^{-1}\cdot(s\tilde{b}_{K_1}(s,t))\cdot\ldots\cdot(s\tilde{b}_{K_m}(s,t))$. Substituting $s=-1$ and using formula (8.5) gives the required relation. □

PROPOSITION 8.5. *For any nonempty complexes K and L there holds*

$$q_{K(L,\dots,L)}(t) = q_K(q_L(t)).$$

PROOF. By corollary 7.7 $\tilde{b}_{K(L,\dots,L)}(s,t) = \tilde{\beta}_K(s^{-1}, s\tilde{b}_L(s,t),\dots,s\tilde{b}_L(s,t))$. Substituting $s=-1$ gives

$$-q_{K(L,\dots,L)}(t^2) = \tilde{\beta}_K(-1, q_L(t^2),\dots,q_L(t^2)) = -q_K(q_L(t^2))$$

which was to be proved. □

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